

THE STRONG "ZERO-TWO" LAW FOR POSITIVE CONTRACTIONS OF BANACH-KANTOROVICH L_p -LATTICES

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ABSTRACT. In the present paper we study majorizable operators acting on Banach-Kantorovich L_p -lattices, constructed by a measure m with values in the ring of all measurable functions. Then using methods of measurable bundles of Banach-Kantorovich lattices, we prove the strong "zero-two" law for positive contractions of the Banach-Kantorovich L_p -lattices.

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1. INTRODUCTION

Starting from von Neumann's [20] pioneering work, the development of the theory of Banach bundles had been stimulated by many works (see for example [13, 14]). There are many papers devoted to the applications of this theory to several branches of analysis [1, 16, 17, 23]. Moreover, this theory is well-connected with the theory of vector-valued Banach spaces [12, 13], which has several applications (see for example, [18]). In the present paper, we concentrate ourselves to the theory of Banach bundles of L_0 -valued Banach spaces (see for more details [6, 13]). Note that such spaces are called *Banach-Kantorovich spaces*. In [13, 14, 17]) the theory of Banach-Kantorovich spaces were developed. It is known [13] that the theory of measurable bundles of Banach lattices is sufficiently well explored. Therefore, it is natural to employ methods of measurable bundles of such spaces to investigate functional properties of Banach-Kantorovich spaces. It is an effective tool which gives a good opportunity to obtain various properties of these spaces [4, 5]. For example, in [7, 6] Banach-Kantorovich lattice $L_p(\nabla, \mu)$ is represented as a measurable bundle of classical L_p -lattices. Naturally, these functional Banach-Kantorovich spaces have many similar properties like the classical ones, constructed by the real valued measures. In [2, 10] this allowed to establish several weighted ergodic theorems for positive contractions of $L_p(\nabla, \mu)$ -spaces. In [5] the convergence theorems of martingales on such lattices has been proved. Some other applications of the measurable bundles of Banach-Kantorovich spaces can be found in [1, 11].

In [19] Ornstein and Sucheston proved that, for any positive contraction T on an L_1 -space, one has either $\|T^n - T^{n+1}\|_1 = 2$ for all n or $\lim_{n \rightarrow \infty} \|T^n - T^{n+1}\|_1 = 0$. An extension of this result to positive operators on L^∞ -spaces was given by Foguel [3]. In [24] Zahoropol generalized these results, called "zero-two" laws, and his result can be formulated as follows:

Theorem 1.1. *Let T be a positive contraction of L_p , $p > 1, p \neq 2$. If the following relation holds $\|T^{m+1} - T^m\| < 2$ for some $m \in \mathbb{N} \cup \{0\}$, then*

$$\lim_{n \rightarrow \infty} \|T^{n+1} - T^n\| = 0.$$

In [15] this theorem was established for Köthe spaces. In particular, from that result it follows the statement of the theorem for a case $p = 2$.

Furthermore, the strong "zero-two" law for positive contractions of L_p -spaces, $1 \leq p < +\infty$ was proved in [22]. This result is formulated as follows:

Theorem 1.2. *Let $1 \leq p < +\infty$ and T be a positive contraction of L_p . If $\|T^{m+1} - T^m\| < 2$ for some $m \in \mathbb{N} \cup \{0\}$, then*

$$\lim_{n \rightarrow \infty} \|T^{n+1} - T^n\| = 0.$$

In [9] we have generalized Theorem 1.1 for the positive contractions of the Banach-Kantorovich L_p -lattices. Namely, the following result was proved.

Theorem 1.3. *Let $T : L_p(\nabla, m) \rightarrow L_p(\nabla, m)$, $p > 1, p \neq 2$ be a positive linear contraction such that $T\mathbf{1} \leq \mathbf{1}$. If one had $\|T^{m+1} - T^m\| < 2 \cdot \mathbf{1}$ for some $m \in \mathbb{N} \cup \{0\}$. Then*

$$(o) - \lim_{n \rightarrow \infty} \|T^{n+1} - T^n\| = 0.$$

The main aim of this paper is to prove the strong "zero-two" law for the positive contractions of the Banach-Kantorovich lattices $L_p(\nabla, m)$. To establish the main aim, we first study majorizable operators acting on Banach-Kantorovich L_p -lattices (see Section 3). Then using methods of measurable bundles of Banach-Kantorovich lattices, in section 4 we prove the main result of the present paper.

2. PRELIMINARIES

Let (Ω, Σ, μ) be a complete measure space with a finite measure μ . By $\mathcal{L}(\Omega)$ (resp. $\mathcal{L}_\infty(\Omega)$) we denote the set of all (resp. essentially bounded) measurable real functions defined on Ω a.e. By the standard way, we introduce an equivalence relation on $\mathcal{L}(\Omega)$ by putting $f \sim g$ whenever $f = g$ a.e. The set $L_0(\Omega)$ of all cosets $f^\sim = \{g \in \mathcal{L}(\Omega) : f \sim g\}$, endowed with the natural algebraic operations, is an algebra with unit $\mathbf{1}(\omega) = 1$ over the field of reals \mathbb{R} . Moreover, with respect to the partial order $f^\sim \leq g^\sim \Leftrightarrow f \leq g$ a.e., the algebra $L_0(\Omega)$ is a Dedekind complete Riesz space with weak unit $\mathbf{1}$, and the set $B(\Omega) := B(\Omega, \Sigma, \mu)$ of all idempotents in $L_0(\Omega)$ is a complete Boolean algebra. Furthermore, $L_\infty(\Omega) = \{f^\sim : f \in \mathcal{L}_\infty(\Omega)\}$ is an order ideal in $L_0(\Omega)$ generated by $\mathbf{1}$. In what follows, we will write $f \in L_0(\Omega)$ instead of $f^\sim \in L_0(\Omega)$ by assuming that the coset of f is considered.

Let E be a linear space over the real field \mathbb{R} . By $\|\cdot\|$ we denote a $L_0(\Omega)$ -valued norm on E . Then the pair $(E, \|\cdot\|)$ is called a *lattice-normed space (LNS) over $L_0(\Omega)$* . An LNS E is said to be *d-decomposable* if for every $x \in E$ and the decomposition $\|x\| = f + g$ with f and g disjoint positive elements in $L_0(\Omega)$ there exist $y, z \in E$ such that $x = y + z$ with $\|y\| = f$, $\|z\| = g$.

Suppose that $(E, \|\cdot\|)$ is an LNS over $L_0(\Omega)$. A net $\{x_\alpha\}$ of elements of E is said to be *(bo)-converging* to $x \in E$ (in this case we write $x = (bo)\text{-}\lim x_\alpha$), if the net $\{\|x_\alpha - x\|\}$ *(o)-converges* to zero (here *(o)-convergence* means the order convergence) in $L_0(\Omega)$ (written as $(o)\text{-}\lim \|x_\alpha - x\| = 0$). A net $\{x_\alpha\}_{\alpha \in A}$ is called *(bo)-fundamental* if $(x_\alpha - x_\beta)_{(\alpha, \beta) \in A \times A}$ *(bo)-converges* to zero.

An LNS in which every *(bo)-fundamental* net *(bo)-converges* is called *(bo)-complete*. A *Banach-Kantorovich space (BKS) over $L_0(\Omega)$* is a *(bo)-complete d-decomposable* LNS over $L_0(\Omega)$. It is well known [16],[17] that every BKS E over $L_0(\Omega)$ admits an $L_0(\Omega)$ -module structure such that $\|fx\| = |f| \cdot \|x\|$ for every $x \in E$, $f \in L_0(\Omega)$, where $|f|$ is the modulus of a

function $f \in L_0(\Omega)$. A BKS $(\mathcal{U}, \|\cdot\|)$ is called a *Banach-Kantorovich lattice* if \mathcal{U} is a vector lattice and the norm $\|\cdot\|$ is monotone, i.e. $|u_1| \leq |u_2|$ implies $\|u_1\| \leq \|u_2\|$. It is known [16] that the cone \mathcal{U}_+ of positive elements is (bo)-closed.

Let ∇ be an arbitrary complete Boolean algebra and let $X(\nabla)$ be the Stone space of ∇ . Assume that $L_0(\nabla) := C_\infty(X(\nabla))$ be the algebra of all continuous functions $x : X(\nabla) \rightarrow [-\infty, +\infty]$ that take the values $\pm\infty$ only on nowhere dense subsets of $X(\nabla)$. Finally, by $C(X(\nabla))$ we denote the subalgebra of all continuous real functions on $X(\nabla)$.

Given a complete Boolean algebra ∇ , let us consider a mapping $m : \nabla \rightarrow L_0(\Omega)$. Such a mapping is called a $L_0(\Omega)$ -valued measure if one has

- (i) $m(e) \geq 0$ for all $e \in \nabla$ and $m(e) = 0 \Leftrightarrow e = 0$;
- (ii) $m(e \vee g) = m(e) + m(g)$ if $e \wedge g = 0, e, g \in \nabla$;
- (iii) $m(e_\alpha) \downarrow 0$ for any net $e_\alpha \downarrow 0$.

Following the well-known scheme of the construction of L_p -spaces, a space $L_p(\nabla, m)$ can be defined by

$$L_p(\nabla, m) = \left\{ f \in L_0(\nabla) : |f|_p := \int |f|^p dm - \text{exist} \right\}, \quad p \geq 1$$

where m is a $L_0(\Omega)$ -valued measure on ∇ .

A $L_0(\Omega)$ -valued measure m is said to be *disjunctive decomposable* (*d-decomposable*), if for every $e \in \nabla$ and the decomposition $m(e) = a_1 + a_2$, $a_1 \wedge a_2 = 0$, $a_i \in L_0(B)$ there exit $e_1, e_2 \in \nabla$ such that $e = e_1 \vee e_2$ and $m(e_i) = a_i, i = 1, 2$.

Theorem 2.1. [6] *The following statements hold:*

- (i) *The pair $(L_p(\nabla, m), |\cdot|_p)$ is (bo)-complete lattice. Moreover, it is an ideal linear subspace of $L_0(\nabla)$, i.e. from $|x| \leq |y|$, $y \in L_p(\nabla, m)$, $x \in L_0(\nabla)$ it follows that $x \in L_p(\nabla, m)$ and $|x|_p \leq |y|_p$;*
- (ii) *If $0 \leq x_\alpha \in L_p(\nabla, m)$ and $x_\alpha \downarrow 0$, then $|x_\alpha|_p \downarrow 0$;*
- (iii) *If the measure m is d-decomposable, then $|\alpha|x|_p = |\alpha||x|_p$ for all $\alpha \in L_0(\Omega), x \in L_p(\nabla, m)$;*
- (iv) *If the measure m is d-decomposable, then $(L_p(\nabla, m), |\cdot|_p)$ is a Banach —Kantorovich space;*
- (v) *One has $L_\infty(\nabla, m) := C(X(\nabla)) \subset L_p(\nabla, m) \subset L_q(\nabla, m)$, $1 \leq q \leq p$. Moreover, $L_\infty(\nabla, m)$ is (bo)-dense in $(L_1(\nabla, m), \|\cdot\|_1)$.*

Now we mention necessary facts from the theory of measurable bundles of Boolean algebras and Banach spaces (see [13] for more details).

Let (Ω, Σ, μ) be the same as above and X be a mapping assigning an L_p -space constructed by a real-valued measure m_ω , i.e. $L_p(\nabla_\omega, m_\omega)$ to each point $\omega \in \Omega$ and let

$$L = \left\{ \sum_{i=1}^n \alpha_i e_i : \alpha_i \in \mathbb{R}, \quad e_i(\omega) \in \nabla_\omega, \quad i = \overline{1, n}, \quad n \in \mathbb{N} \right\}$$

be a set of sections. In [6] it has been established that the pair (X, L) is a measurable bundle of Banach lattices and $L_0(\Omega, X)$ is modulo ordered isomorphic to $L_p(\nabla, \mu)$.

Let ρ be a lifting in $L_\infty(\Omega)$ (see [13]). As before, let ∇ be an arbitrary complete Boolean subalgebra of $\nabla(\Omega)$ and m be an $L_0(\Omega)$ -valued measure on ∇ . By $L_\infty(\nabla, m)$ we denote the set of all essentially bounded functions w.r.t. m taken from $L_0(\nabla)$.

A mapping $\ell : L_\infty(\nabla, m) (\subset L_\infty(\Omega, X)) \rightarrow \mathcal{L}_\infty(\Omega, X)$ is called a *vector-valued lifting* [13] associated with the lifting ρ if it satisfies the following conditions:

- (1) $\ell(\hat{u}) \in \hat{u}$ for all \hat{u} such that $\text{dom}(\hat{u}) = \Omega$;
- (2) $\|\ell(\hat{u})\|_{L_p(\nabla_\omega, m_\omega)} = \rho(|\hat{u}|_p)(\omega)$;
- (3) $\ell(\hat{u} + \hat{v}) = \ell(\hat{u}) + \ell(\hat{v})$ for every $\hat{u}, \hat{v} \in L_\infty(\nabla, m)$;
- (4) $\ell(h \cdot \hat{u}) = \rho(h)\ell(\hat{u})$ for every $\hat{u} \in L_\infty(\nabla, m)$, $h \in L_\infty(\Omega)$;
- (5) $\ell(\hat{u}) \geq 0$ whenever $\hat{u} \geq 0$;
- (6) the set $\{\ell(\hat{u})(\omega) : \hat{u} \in L_\infty(\nabla, m)\}$ is dense in $X(\omega)$ for all $\omega \in \Omega$;
- (7) $\ell(\hat{u} \vee \hat{v}) = \ell(\hat{u}) \vee \ell(\hat{v})$ for every $\hat{u}, \hat{v} \in L_\infty(\nabla, m)$.

In [6] the existence of the vector-valued lifting was proved.

Let $L_p(\nabla, m)$ ($p \geq 1$) be a Banach-Kantorovich lattice. A linear mapping $T : L_p(\nabla, m) \rightarrow L_p(\nabla, m)$ is called *positive* if $T\hat{f} \geq 0$ whenever $\hat{f} \geq 0$. We say that T is a $L_0(\Omega)$ -*bounded mapping* if there exists a function $k \in L_0(\Omega)$ such that $|T\hat{f}|_p \leq k|\hat{f}|_p$ for all $\hat{f} \in L_p(\nabla, \mu)$. For such a mapping we can define an element of $L_0(\Omega)$ as follows

$$\|T\| = \sup_{|\hat{f}|_p \leq 1} |T\hat{f}|_p,$$

which is called an $L_0(\Omega)$ -*valued norm* of T . A mapping T is said to be a *contraction* if one has $\|T\| \leq 1$. Some examples of contractions can be found in [10].

In the sequel we will need the following bundle representation of $L_0(\Omega)$ -linear $L_0(\Omega)$ -bounded operators acting in Banach-Kantorovich lattices.

Theorem 2.2. [9] *Let $L_p(\nabla, m)$ ($p \geq 1$) be a Banach-Kantorovich lattice, and $L_p(\nabla_\omega, m_\omega)$ be the corresponding L_p -spaces constructed by real valued measures. Let $T : L_p(\nabla, m) \rightarrow L_p(\nabla, m)$ be a positive linear contraction such that $T\mathbf{1} \leq \mathbf{1}$. Then for every $\omega \in \Omega$ there exists a positive contraction $T_\omega : L_p(\nabla_\omega, \mu_\omega) \rightarrow L_p(\nabla_\omega, m_\omega)$ such that $T_\omega f(\omega) = (T\hat{f})(\omega)$ a.e. for every $\hat{f} \in L_p(\nabla, m)$.*

3. MAJORIZABLE OPERATORS IN BANACH-KANTOROVICH L_p -LATTICES

In this section, we are going to study majorizable operators in Banach-Kantorovich L_p -lattices.

Theorem 3.1. *Let $T : L_1(\nabla, m) \rightarrow L_1(\nabla, m)$ be an $L_0(\Omega)$ -bounded linear operator in Banach-Kantorovich lattice $L_1(\nabla, m)$. Then there exists a unique $|T|$ - $L_0(\Omega)$ -bounded linear operator in $L_1(\nabla, m)$ such that*

- (a) $\|T\| = \||T|\|$;
- (b) one has $|T\hat{f}| \leq |T||\hat{f}|$, for all $\hat{f} \in L_1(\nabla, m)$;
- (c) for each $\hat{f} \in L_1(\nabla, m)$ with $\hat{f} \geq 0$ one has $|T|\hat{f} = \sup\{|T\hat{g}| : \hat{g} \in L_1(\nabla, m), |\hat{g}| \leq \hat{f}\}$;
- (d) $\|T\|_\infty = \||T|\|_\infty$.

Proof. Let \mathcal{P} denote the family of all finite measurable partitions $\pi = \{B_1, B_2, \dots, B_m\}$ of Ω . We partially order \mathcal{P} in the usual way, i.e. for $\pi = \{B_1, B_2, \dots, B_m\}$ and $\pi' = \{B'_1, B'_2, \dots, B'_k\}$ we write $\pi \leq \pi'$ if π' is a refinement of π , i.e. each set B_i is a union of sets $\{B'_i\}$.

Given $\pi \in \mathcal{P}$, and for every $\hat{f} \in L_1(\nabla, m)$, $\hat{f} \geq 0$ we define

$$T_\pi \hat{f} = \sum_{i=1}^m |T(\chi_{B_i} \hat{f})|.$$

Clearly $\pi \leq \pi'$ implies $T_\pi \hat{f} \leq T_{\pi'} \hat{f}$. From $|\hat{f}|_1 = \sum_{i=1}^m |\chi_{B_i} \hat{f}|_1$ we obtain $|T_\pi \hat{f}|_1 \leq \|T\| |\hat{f}|_1$. Since $\{T_\pi \hat{f} : \pi \in \mathcal{P}\}$ is increasing on \mathcal{P} and is norm bounded, therefore one can define

$$|T|\hat{f} := \lim_{\pi \in \mathcal{P}} T_\pi \hat{f}, \quad \hat{f} \geq 0.$$

We clearly have

$$(3.1) \quad \| |T|\hat{f} \|_1 \leq \|T\| \|\hat{f}\|_1, \quad \hat{f} \geq 0$$

and $|T|$ is linear on positive functions. Therefore $|T|$ can be extended by the linearity to whole $L_1(\nabla, m)$. This extension is again denoted by $|T|$.

For $\hat{f} \geq 0$ and $|\hat{g}| \leq \hat{f}$ we obtain $|T|\hat{f} \geq |T|\hat{g}$ by means of the approximation argument with simple functions. This yields (b).

(c). From (b) we have $|T|\hat{g} \geq |T|\hat{g}|$, i.e. T has a positive majorant. Then by [21, Theorem VIII 1.1] T is regular. Hence, using [21, formula (10), p.231] one finds $|T|\hat{f} = \sup\{|T|\hat{g}| : \hat{g} \in L_1(\nabla, m), |\hat{g}| \leq \hat{f}\}$.

(a). Again from (b) we get $\|T\| \leq \| |T| \|$ and by (3.1) one finds $\| |T| \| \leq \|T\|$. Hence, $\|T\| = \| |T| \|$.

(d). Let $\hat{f} \in L^\infty(\hat{\nabla}, \hat{\mu})$. It is then clear that from $|T|\hat{f} \leq \|T\| \hat{f}$ one gets $\|T\|_\infty \|\hat{f}\|_\infty \leq \| |T| \|_\infty \|\hat{f}\|_\infty$ which means $\|T\|_\infty \leq \| |T| \|_\infty$.

Using (c) we obtain

$$|T|\hat{f} = \sup_{|\hat{g}| \leq \hat{f}} |T|\hat{g} \leq \sup_{|\hat{g}| \leq \hat{f}} \|T\|_\infty \|\hat{g}\|_\infty \mathbf{1} \leq \|T\|_\infty \|\hat{f}\|_\infty \mathbf{1}.$$

Hence, $\| |T| \|_\infty \leq \|T\|_\infty$ and $\| |T| \|_\infty = \|T\|_\infty$. □

Definition 3.2. A linear operator $A : L_p(\nabla, m) \rightarrow L_p(\nabla, m)$ is called majorizable if there exists an $L_0(\Omega)$ - bounded positive linear operator $S : L_p(\nabla, m) \rightarrow L_p(\nabla, m)$ such that

$$|A\hat{f}| \leq S(|\hat{f}|)$$

for all $\hat{f} \in L_p(\nabla, m)$. The operator S is called majorant.

Theorem 3.3. Let $T : L_p(\nabla, m) \rightarrow L_p(\nabla, m)$ be a majorizable operator with a majorant S on Banach-Kantorovich lattice $L_p(\nabla, m)$. Then there exists a unique $|T|$ - $L_0(\Omega)$ - bounded linear operator on $L_p(\nabla, m)$ such that

- (a) $\| |T| \| \leq \|S\|$;
- (b) one has $|T|\hat{f} \leq |T|\hat{f}|$, for all $\hat{f} \in L_p(\nabla, m)$;
- (c) for each $\hat{f} \in L_p(\nabla, m)$, $\hat{f} \geq 0$ one has $|T|\hat{f} = \sup\{|T|\hat{g}| : \hat{g} \in L_p(\nabla, m), |\hat{g}| \leq \hat{f}\}$;

Proof. The proof of the existence of $|T|$ and (b), (c) are similar to the proof of Theorem 3.1. Now we prove (a). From

$$|T|\hat{f} = \sup\{|T\hat{g}| : \hat{g} \in L_p(\nabla, m), |\hat{g}| \leq \hat{f}\} \leq \sup\{S|\hat{g}| : \hat{g} \in L_p(\nabla, m), |\hat{g}| \leq \hat{f}\} = S\hat{f}$$

we get

$$\| |T|\hat{f} \|_p \leq \| S\hat{f} \|_p \leq \| S \| \| \hat{f} \|_p$$

hence

$$\| |T| \| \leq \| S \|.$$

This completes the proof. \square

Theorem 3.4. *If $A : L_p(\nabla, m) \rightarrow L_p(\nabla, m)$ is a majorizable operator, and its majorant S is a contraction with $S\mathbf{1} \leq \mathbf{1}$, then for every $\omega \in \Omega$ there exists a majorizable operator $A_\omega : L_p(\nabla_\omega, m_\omega) \rightarrow L_p(\nabla_\omega, m_\omega)$ such that*

$$A_\omega f(\omega) = (A\hat{f})(\omega) \quad a.e.$$

for all $\hat{f} \in L_p(\nabla, m)$.

Proof. Since S is a contraction and $S\mathbf{1} \leq \mathbf{1}$, we obtain that $A(L_\infty(\nabla, m)) \subset L_\infty(\nabla, m)$.

Now we define a linear operator φ_ω from $\{\ell(\hat{f})(\omega) : \hat{f} \in L_\infty(\nabla, m)\}$ into $L_p(\nabla_\omega, m_\omega)$ by

$$\varphi_\omega(\ell(\hat{f})(\omega)) = \ell(A\hat{f})(\omega)$$

where ℓ is the vector lifting of $L_\infty(\nabla, m)$ associated with the lifting ρ .

From the majorizability of A one gets

$$|\varphi_\omega(\ell(\hat{f})(\omega))| = |\ell(A\hat{f})(\omega)| = \ell(|A\hat{f}|)(\omega) \leq \ell(S|\hat{f}|)(\omega) = S'_\omega(\ell(|\hat{f}|)(\omega)) = S'_\omega(|\ell(\hat{f})|(\omega))$$

for any positive $\hat{f} \in L_\infty(\nabla, m)$, where S'_ω is a positive contraction on $\{\ell(\hat{f})(\omega) : \hat{f} \in L_\infty(\nabla, m)\}$. This means that φ_ω is a majorizable operator on $\{\ell(\hat{f})(\omega) : \hat{f} \in L_\infty(\nabla, m)\}$.

From $|S\hat{f}|_p \leq |\hat{f}|_p$ we obtain

$$\|\ell(A\hat{f})(\omega)\|_{L_p(\nabla_\omega, m_\omega)} = \rho(|A\hat{f}|_p)(\omega) \leq \rho(|S\hat{f}|_p)(\omega) \leq \rho(|\hat{f}|_p)(\omega) = \|\ell(\hat{f})(\omega)\|_{L_p(\nabla_\omega, m_\omega)}$$

which implies that φ_ω and S'_ω are well defined and bounded. Moreover, S'_ω is positive (see Theorem 2.2). Due to the density of $\{\ell(\hat{f})(\omega) : \hat{f} \in L_\infty(\nabla, m)\}$ in $L_p(\nabla_\omega, m_\omega)$, we can extend φ_ω and S'_ω , respectively, to $L_p(\nabla_\omega, m_\omega)$. We respectively denote the extensions by A_ω and S_ω . One can see that A_ω is bounded, and S_ω is positive bounded.

From

$$|\varphi_\omega(\ell(\hat{f})(\omega))| \leq S'_\omega(|\ell(\hat{f})(\omega)|)$$

for any $\hat{f} \in L_\infty(\nabla, m)$ one finds

$$|A_\omega(f(\omega))| \leq S_\omega(|f(\omega)|)$$

i.e. A_ω is majorizable.

Repeating the argument of the proof of [9, Theorem 2.1], we can prove that

$$A_\omega f(\omega) = (A\hat{f})(\omega)$$

for almost all $\omega \in \Omega$ and for all $\hat{f} \in L_p(\nabla, m)$. This completes the proof. \square

Theorem 3.5. *If $A : L_p(\nabla, m) \rightarrow L_p(\nabla, m)$ is a majorizable operator, and its majorant S is a contraction with $S\mathbf{1} \leq \mathbf{1}$, then*

$$\| |A|_\omega \|_{p,\omega} = \| |A_\omega| \|_{p,\omega}$$

for almost all $\omega \in \Omega$, where $\| \cdot \|_{p,\omega}$ is the norm of an operator from $L_p(\nabla_\omega, m_\omega)$ to $L_p(\nabla_\omega, m_\omega)$.

Proof. Due to $-|A| \leq A \leq |A|$ we have $-|A|_\omega \leq A_\omega \leq |A|_\omega$ which yields $|A_\omega| \leq |A|_\omega$ for almost all $\omega \in \Omega$. Hence, $\| |A|_\omega \|_{p,\omega} \geq \| |A_\omega| \|_{p,\omega}$ for almost all $\omega \in \Omega$.

Let $\{\pi_n\}$ be an increasing sequence in \mathcal{P} such that $|A|\hat{f} = (bo) - \lim_{n \rightarrow \infty} A_{\pi_n}\hat{f}$, for $0 \leq \hat{f} \in L_p(\nabla, m)$.

One can see that

$$(3.2) \quad (A_{\pi_n}\hat{f})(\omega) = \sum_{i=1}^m |A(\chi_{B_i}\hat{f})|(\omega) = \sum_{i=1}^m |A_\omega(\chi_{B_i}(\omega)f)(\omega)| = A_{\omega,\pi_n}f(\omega)$$

for almost all $\omega \in \Omega$.

Now using

$$|A|\hat{f} = (bo) - \lim_{n \rightarrow \infty} A_{\pi_n}\hat{f} \text{ in } L_p(\nabla, m),$$

with (3.2) we obtain $|A_{\pi_n}\hat{f}|_p \xrightarrow{(o)} | |A|\hat{f} |_p$ or $|A_{\pi_n}\hat{f}|_p(\omega) \rightarrow | |A|\hat{f} |_p(\omega)$ for almost all $\omega \in \Omega$. Hence,

$$\| A_{\pi_n,\omega}f(\omega) \|_{L_p(\nabla_\omega, m_\omega)} \rightarrow \| |A|_\omega f(\omega) \|_{L_p(\nabla_\omega, m_\omega)}$$

for almost all $\omega \in \Omega$.

On the other hand, one has

$$\lim_{n \rightarrow \infty} \| A_{\pi_n,\omega}f(\omega) \|_{L_p(\nabla_\omega, m_\omega)} \leq \| |A_\omega| f(\omega) \|_{L_p(\nabla_\omega, m_\omega)}$$

for almost all $\omega \in \Omega$. This means that

$$\| |A|_\omega f(\omega) \|_{L_p(\nabla_\omega, m_\omega)} \leq \| |A_\omega| f(\omega) \|_{L_p(\nabla_\omega, m_\omega)}$$

or

$$\| |A|_\omega \|_{p,\omega} \leq \| |A_\omega| \|_{p,\omega}$$

for almost all $\omega \in \Omega$. Hence

$$\| |A|_\omega \|_{p,\omega} = \| |A_\omega| \|_{p,\omega}$$

for almost all $\omega \in \Omega$. This completes the proof. \square

4. THE STRONG "ZERO-TWO" LAW

In this section we are going to prove an analog of the strong "zero-two" law for positive contractions in the Banach-Kantorovich L_p -lattices. Before the formulation of the main result, we need some auxiliary results.

Proposition 4.1. *Let $T, S : L_p(\nabla, m) \rightarrow L_p(\nabla, m)$ be two positive linear contractions such that $T\mathbf{1} \leq \mathbf{1}$, $S\mathbf{1} \leq \mathbf{1}$. Then*

$$\| |T_\omega - S_\omega| \|_{p,\omega} \geq \| |T - S| \|(\omega), \quad a.e.$$

here $|\cdot|$ means the modulus of an operator.

Proof. Due to $(T - S)(\hat{f}) \leq T(\hat{f})$ for any positive $\hat{f} \in L_p(\nabla, m)$ one gets

$$|(T - S)(\hat{f})| \leq T(|\hat{f}|)$$

for any $\hat{f} \in L_p(\nabla, m)$. Hence $T - S$ is majorizable. Since T is a contraction and $T\mathbf{1} \leq \mathbf{1}$ by Theorem 3.5, we obtain $\| |T - S|_\omega \|_{p, \omega} = \| |T_\omega - S_\omega| \|_{p, \omega}$ for almost all $\omega \in \Omega$. By [8, Proposition 2] for any $\varepsilon > 0$ there exists $\hat{f} \in L_p(\nabla, m)$ with $|\hat{f}|_p = \mathbf{1}$ such that

$$\| |T - S| \| - \varepsilon \mathbf{1} \leq |T - S|\hat{f}|_p.$$

Then

$$\begin{aligned} \| |T - S| \|(\omega) - \varepsilon \mathbf{1} &\leq |T - S|\hat{f}|_p(\omega) = \| (|T - S|\hat{f})(\omega) \|_{L_p(\nabla_\omega, m_\omega)} \\ &= \| |T - S|_\omega \hat{f}(\omega) \|_{L_p(\nabla_\omega, m_\omega)} \leq \| |T - S|_\omega \|_{p, \omega} \\ &= \| |T_\omega - S_\omega| \|_{p, \omega} \end{aligned}$$

for almost all $\omega \in \Omega$. The arbitrariness of $\varepsilon > 0$ implies the statement. \square

Corollary 4.2. *Let $T, S : L_p(\nabla, m) \rightarrow L_p(\nabla, m)$ be two positive linear contractions such that $T\mathbf{1} \leq \mathbf{1}, S\mathbf{1} \leq \mathbf{1}$. Then*

$$\| |T_\omega - S_\omega| \|_{p, \omega} = \| |T - S| \|(\omega), \quad a.e.$$

The proof follows from [9, Proposition 3.2] and Proposition 4.1.

The next theorem is our main result of the present paper.

Theorem 4.3. *Let $T : L_p(\nabla, m) \rightarrow L_p(\nabla, m)$ be a positive linear contraction such that $T\mathbf{1} \leq \mathbf{1}$. If one has $\| |T^{m+1} - T^m| \| < 2 \cdot \mathbf{1}$ for some $m \in \mathbb{N} \cup \{0\}$. Then*

$$(o) - \lim_{n \rightarrow \infty} \| |T^{n+1} - T^n| \| = 0.$$

Proof. From Corollary 4.2 it follows that

$$\| |T_\omega^{m+1} - T_\omega^m| \|_{p, \omega} = \| |T^{m+1} - T^m| \|(\omega), \quad a.e.$$

on Ω . Therefore, due to $\| |T^{m+1} - T^m| \| < 2 \cdot \mathbf{1}$ for some $m \in \mathbb{N} \cup \{0\}$ we find $\| |T_\omega^{m+1} - T_\omega^m| \|_{p, \omega} < 2$ for almost all $\omega \in \Omega$. According to Theorem 2.2 we conclude that T_ω is a positive contraction on $L_p(\nabla_\omega, m_\omega)$. Hence, the contraction T_ω satisfies the conditions of Theorem 1.2 for almost all $\omega \in \Omega$, which yields that

$$\lim_{n \rightarrow \infty} \| |T_\omega^{n+1} - T_\omega^n| \| = 0$$

for almost all $\omega \in \Omega$. Then again using Corollary 4.2 we obtain

$$\lim_{n \rightarrow \infty} \| |T^{n+1} - T^n| \|(\omega) = 0$$

for almost all $\omega \in \Omega$. Therefore,

$$(o) - \lim_{n \rightarrow \infty} \| |T^{n+1} - T^n| \| = 0.$$

This completes the proof. \square

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